ALGEBRA MID TERMINAL EXAMINATION

This exam is of **30 marks** and is **2 hours long**. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly. Please sign the following statement:

I have not used any unfair or illegal means to answer any of the questions in this exam.

Name: **Signature:**

1. Let R be a commutative ring with 1. Let S be a multiplicative set such that $0 \notin S$. Let I be an ideal which is maximal with respect to the condition that $S \cap I = \emptyset$. Show that I is a prime ideal. (4)

2. Let R be a commutative ring with identity. Let $f = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$ be a polynomial in $R[X]$. Show that (10)

$$f \text{ is nilpotent} \Leftrightarrow a_i \text{ are nilpotent for all } i$$

3a. Let R be a commutative ring with identity. Show that $f(X)$ is irreducible $\Leftrightarrow$ $f(X+c)$ is irreducible for any $c \in R$. (4)

3b. Let p be a prime number. Show that the polynomial

$$\Phi_p(X) = X^{p-1} + X^{p-2} + \cdots + 1$$

is irreducible in $\mathbb{Z}[X]$. (4)

4. Let R be a Euclidean ring with Euclidean function ϕ. Show

a. There exists an identity 1_R in R. (4)

b. u is a unit $\Leftrightarrow$ $\phi(u) = \phi(1_R)$. (4)